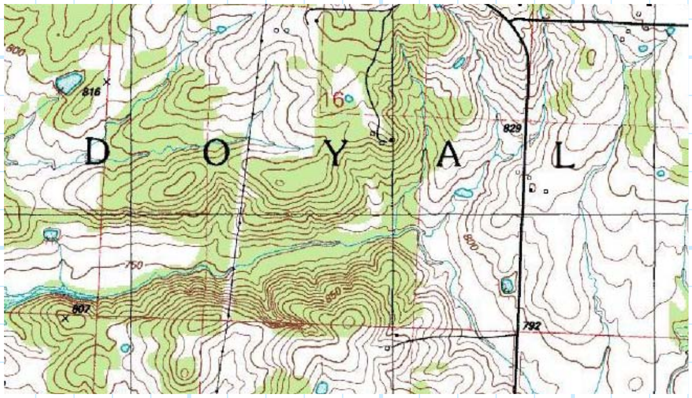
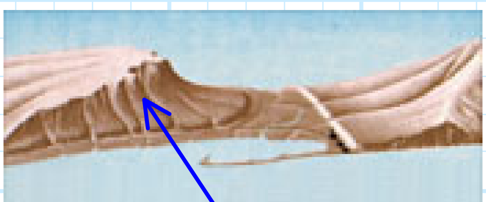
The Gradient

Consider the **topography** of the Earth's surface.



We use contours of constant elevation—called **topographic contours**—to express on maps (a 2-dimensional graphic) the third dimension of elevation (i.e., surface height).

We can infer from these maps the **slope** of the Earth's surface, as topographic contours lie closer together where the surface is very steep.



*See, this indicates the location of a steep and scary **Cliff**!*

From: *erg.usgs.gov/isb/pubs/booklets/symbols/reading.html*

Moreover, we can likewise infer the **direction** of these slopes—a hillside might slope toward the south, or a cliff might drop-off toward the East.

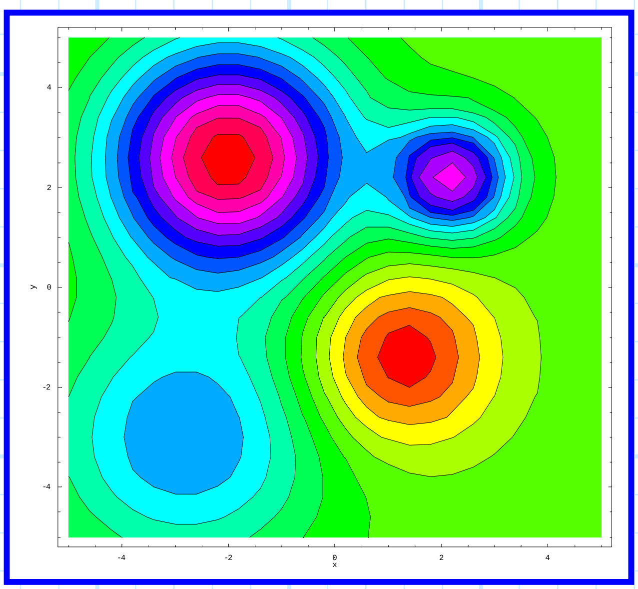
Thus, the slope of the Earth's surface has both a magnitude (e.g., flat or steep) and a direction (e.g. toward the north). In other words, the slope of the Earth's surface is a **vector quantity**!

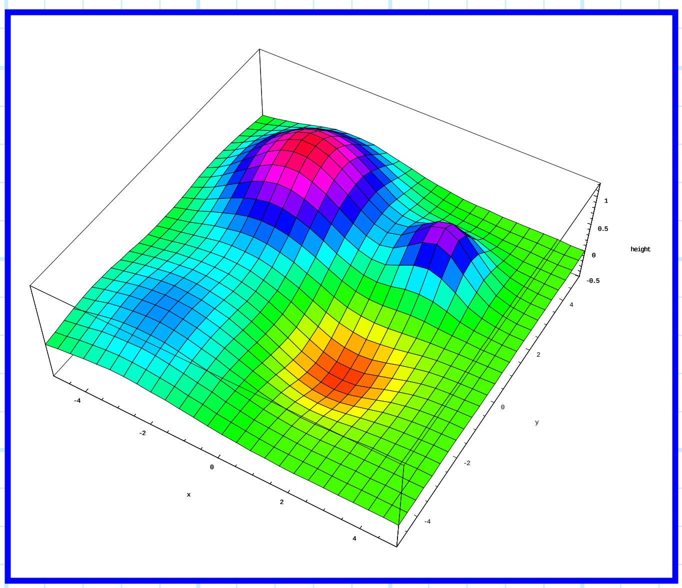
Thus, the surface slope at every point across some section of the Earth (e.g., Douglas County, Colorado, or North America) must be described by a **vector field**!

Q: *Sure, but there isn't any way to calculate this vector field is there?*

A: Yes, there is a very easy way, called the **gradient**.

Say the topography of some small section of the Earth's surface can be described as a **scalar** function $h(x,y)$, where h represents the **height** (elevation) of the Earth at some point denoted by coordinates x and y. E.G.:





Now say we take the **gradient** of scalar field $h(x,y)$. We denote this operation as:

$$\nabla h(\bar{r})$$

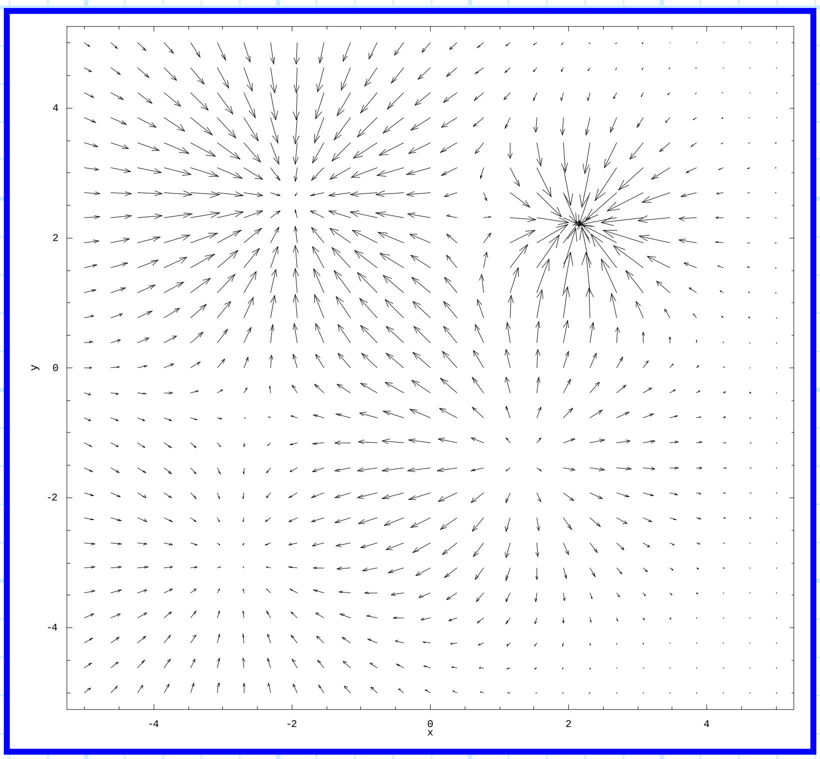
The result of taking the gradient of a scalar field is a **vector field**, i.e.:

$$\nabla h(\bar{r}) = \mathbf{A}(\bar{r})$$

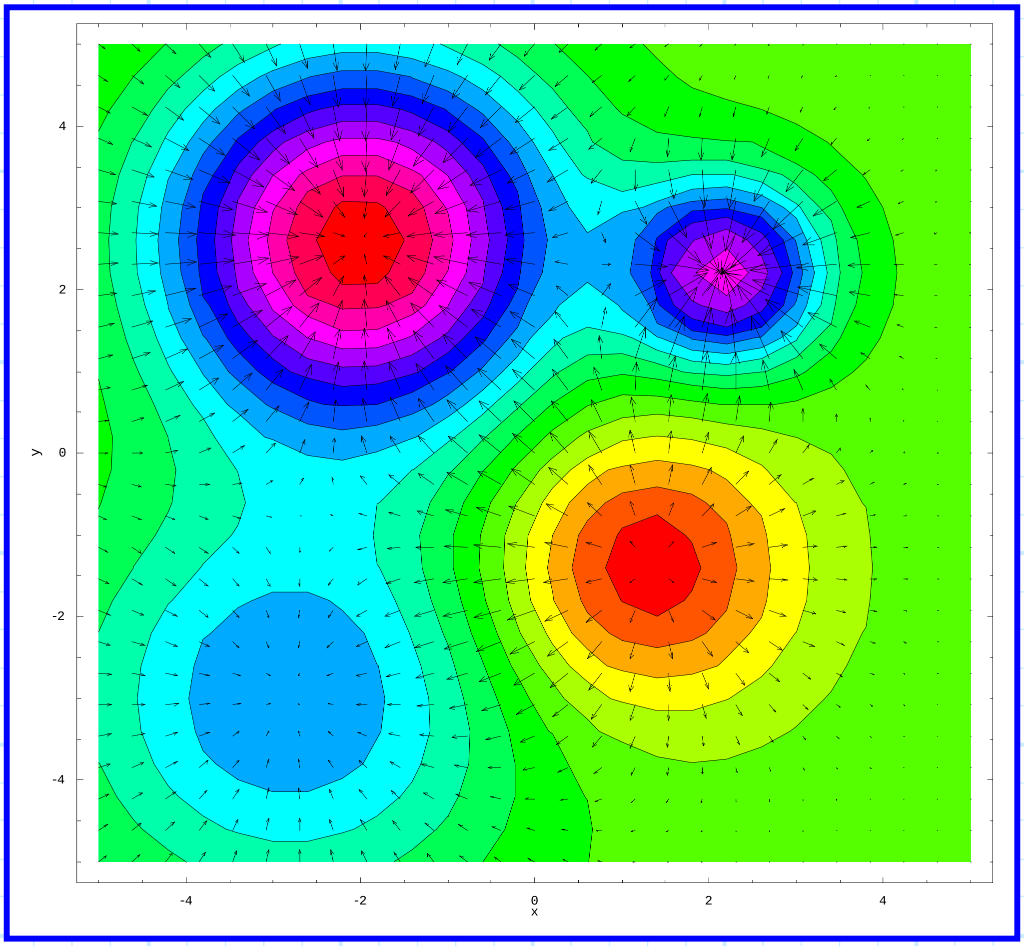
Q: *So just what **is** this resulting vector field, and how does it **relate** to scalar field $h(\bar{r})$??*

For our example here, taking the **gradient** of surface elevation $h(x,y)$ results in the following **vector** field:

???



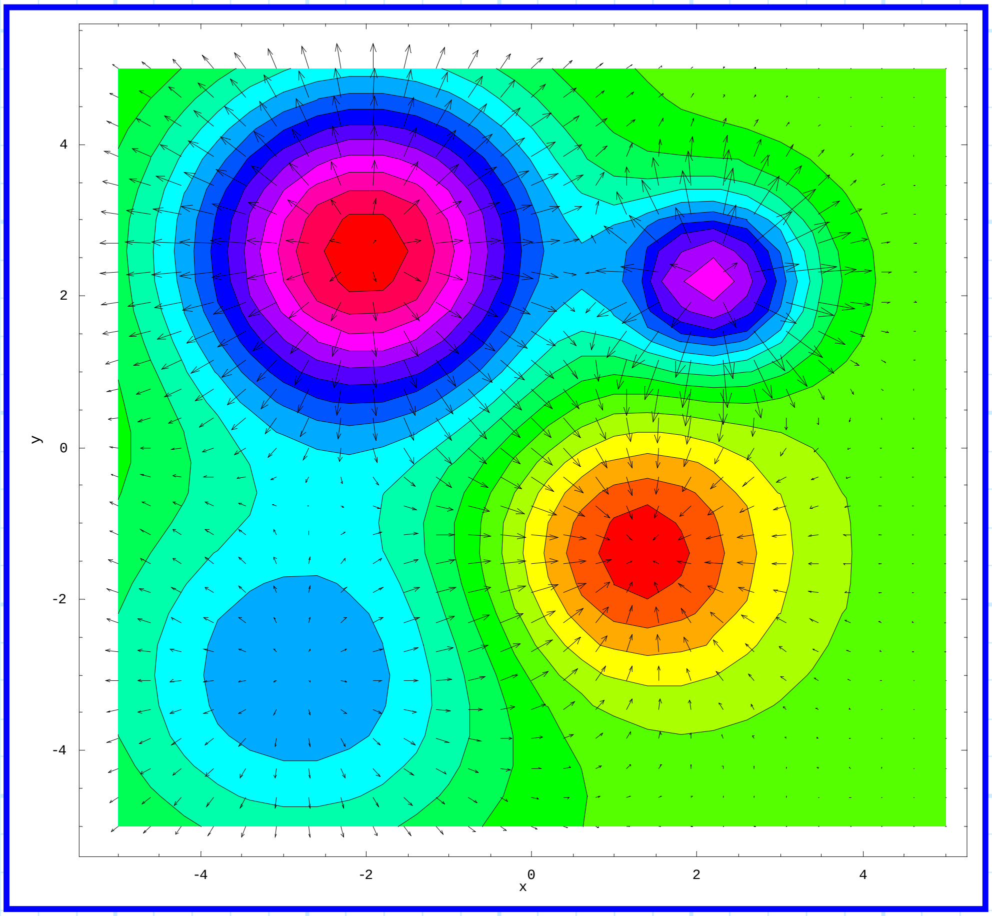
To see how this **vector** field **relates** to the surface height $h(x,y)$, let's place the vector field on top of the topographic plot:



Q: *It appears that the vector field indicates the slope of the surface topology—both its **magnitude** and **direction**!*

A: That's right! The gradient of a **scalar** field provides a **vector** field that states how the scalar value is **changing** throughout space—a change that has both a **magnitude** and **direction**.

It is a bit more "natural" and instructive for our example to examine the **opposite** of the gradient of h(x,y) (i.e., $\mathbf{A}(\bar{r}) = -\nabla h(\bar{r})$). In other words, to plot the vectors such that they are pointing in the "**downhill**" direction.



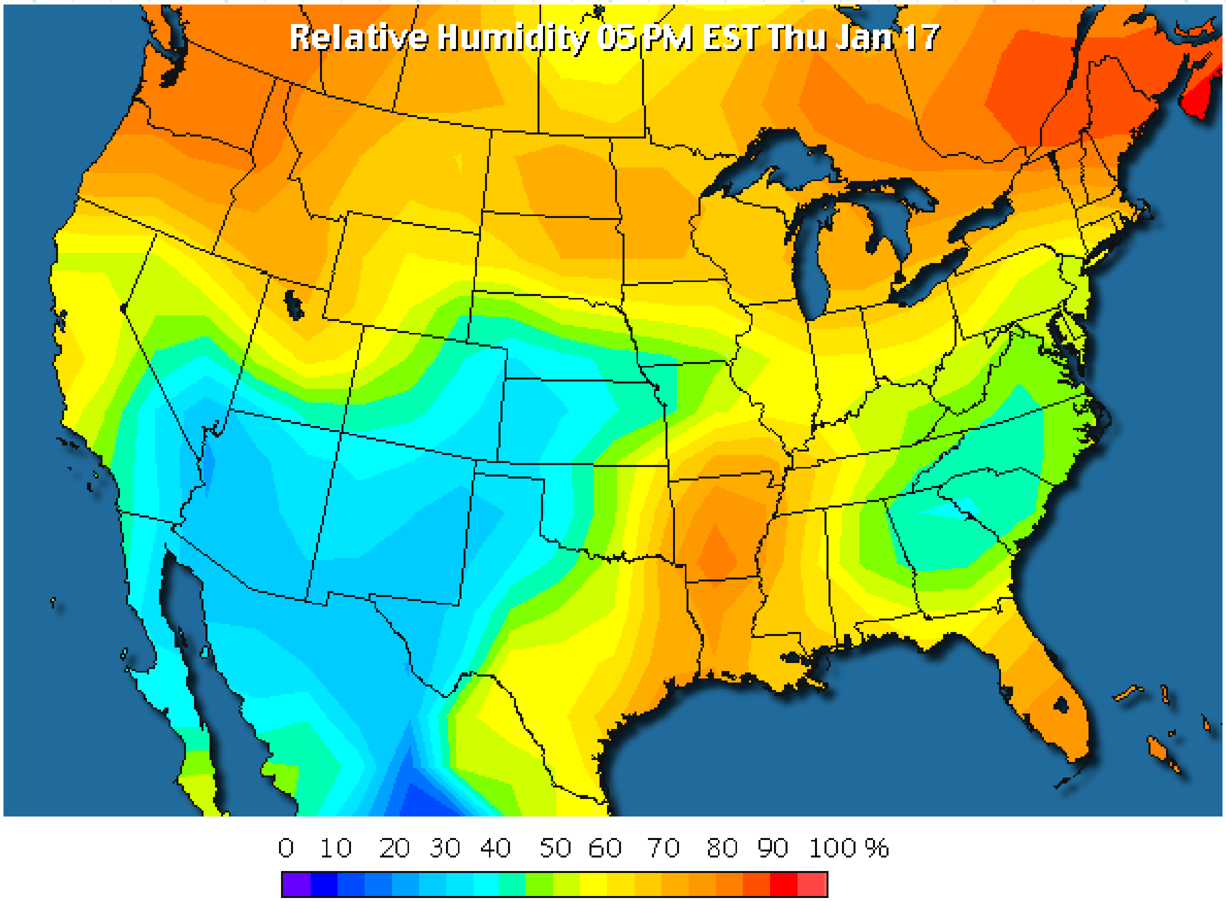
Note these important facts:

* The vectors point in the direction of **maximum change** (i.e., they point straight down the mountain!).

* The vectors always point **orthogonal** to the topographic contours (i.e., the contours of equal surface height).

Now, it is important to understand that the scalar fields we will consider will **not** typically describe the height or altitude of anything! Thus, the slope provided by the gradient is more mathematically "abstract", in the same way we speak about the slope (i.e., derivative) of some curve.

For example, consider the **relative humidity** across the country—a **scalar** function of position.



If we travel in some directions, we will find that the humidity quickly changes. But if we travel in other directions, the humidity will change not at all.

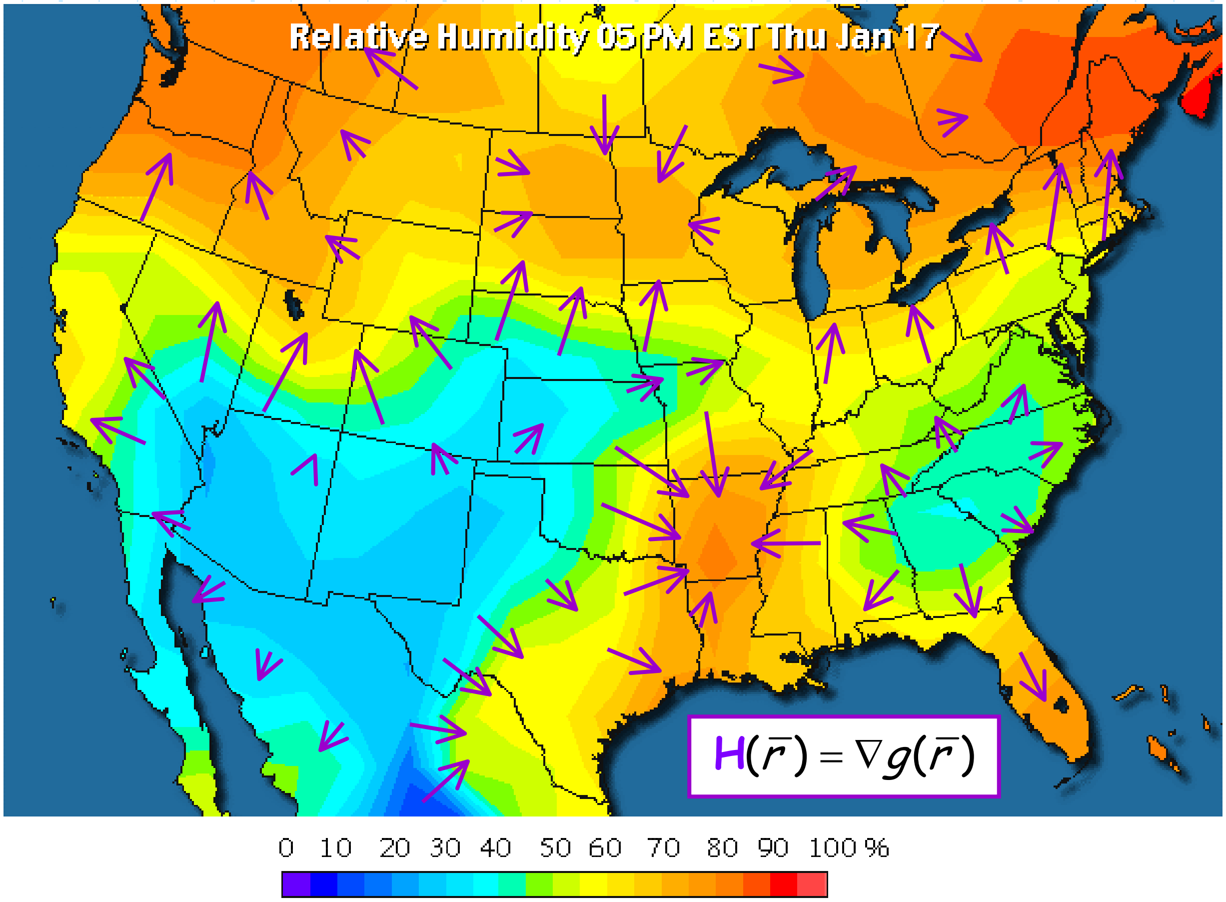
Q: *Say we are located at some point (e.g., Lawrence, KS; Albuquerque, N M; or Ann Arbor, MI), how can we determine the direction where we will experience the greatest change in humidity ?? Also, how can we determine what that change will be ??*

A: The answer to both questions is to take the **gradient** of the scalar field that represents humidity!

If $g(\bar{r})$ is the scalar field that represents the humidity across the country, then we can form a vector field $\mathbf{H}(\bar{r})$ by taking the gradient of $g(\bar{r})$:

$$\mathbf{H}(\bar{r}) = \nabla g(\bar{r})$$

This vector field indicates the **direction** of greatest humidity change (i.e., the direction where the derivative is the largest), as well as the **magnitude** of that change, at every point in the country!



This is likewise true for **any** scalar field. The gradient of a scalar field produces a **vector** field indicating the direction of greatest change (i.e., largest derivative) as well as the magnitude of that change, at every point in space.